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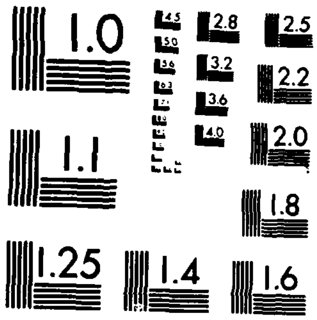

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This paper is concerned with determining a potential $q(x)$ for the steady state Schroedinger equation in two space variables, $x = (x_1, x_2)$:

1.1 $$(\Delta + \omega^2 - q)u = 0 \; .$$

It is assumed there are no bound states.

The data we are given come from the far field scattered by plane waves impinging in a range of directions but measured only in the opposite directions. More precisely, let $p(e,\omega,x)$ be a solution of (1.1) which for $e \cdot x = -\infty$ behaves like $e^{i\omega e\cdot x}/|x|^{\frac{1}{2}}$. Here e is a unit vector.

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We consider the space $\underline{Q}$ of bounded functions also lying in L^1 and assume $q \quad \underline{Q}$.

The solution p of (1.1) is written as $e^{-i\omega e \cdot x} + p_s$ where for $|x| \to \infty$

1.2 $$|x|^{\frac{1}{2}} e^{i\omega|x|} p_s \to s_1(e,\omega,x/|x|), \quad |x| \to \infty ,$$

where s_1 is called the far field. The back scattered field is measured in the direction $-e$ and

1.3 $$s(\omega,e) = s_1(e,\omega,-e) \quad .$$

The problem under study is the determination of q from $s(\omega,e)$. A theorem is proved here that if s is sufficiently small so that the Born approximation is small in norm in the space $\underline{Q}$ there exists a unique solution q to the back scattering problem.

The Inverse Back Scattering Problem for the Schroedinger Equation in Two Space Dimensions

Cathleen S. Morawetz

1. Introduction

This paper is concerned with determining a potential $q(x)$ for the steady state Schroedinger equation in two space variables, $x = (x_1, x_2)$:

$$(\Delta + \omega^2 - q)u = 0 \ . \tag{1.1}$$

It is assumed there are no bound states.

The data ~~we are given~~, come from the far field scattered by plane waves impinging in a range of directions but measured only in the opposite directions. More precisely let $p(e,\omega,x)$ be a solution of (1.1) which for $e\cdot x \to -\infty$ behaves like $e^{-i\omega e\cdot x}$ plus a scattered wave behaving like $e^{i\omega e\cdot x}$. Here e is a unit vector. Clearly this requires some conditions of decay on q.

We consider the space Q of bounded functions also lying in L^1 and assume $q \in Q$. This is a less than perfect space from the point of view of physics since it does not allow for singularities in the potential. However in other work [1] a study is made of an iteration for finding a smooth potential from its back scattered field and this space suffices for that study, see Appendix 2. One would prefer for numerical purposes to consider only compactly supported potentials but the theory does not work for these. It should be noted that the 2D problem is barely admissible for computation as it involves four independent varibles (e,ω,x) and an iteration.

The solution p of (1.1) is written as $e^{-i\omega e\cdot x} + p_s$ where for $|x| \to \infty$

$$|x|^{1/2}\, e^{i\omega|x|}\, p_s \to s_1(e,\omega,x/|x|), \quad |x| \to \infty \ . \tag{1.2}$$

where s_1 is called the far field. The back scattered field is measured in the direction $-e$ and

(1.3) $$s(\omega,e) = s_1(e,\omega,-e) .$$

The problem under study is the determination of q from $s(\omega,e)$.

From the physicist's point of view this may seem an unreasonable restriction on the measurement. At the very least one could measure the scattered field over a range of angles. However this would give us an over-determined and hence intrinsically unstable problem. We would be giving a function of two angles and the frequency ω which is too much data for determining q, a function in polar coordinates of one scalar and one angle. It is reasonable to expect that the linearized theory (the Born approximation) determines how much data should be prescribed. We check this now in 2D. First, rederiving the Born approximation and then proceeding to the main result:

For sufficiently small values of $s(e,\omega)$, there is a unique potential if there are no bound states.

By the obvious linearization, if p_B is the Born approximation to the scattered field p_s then

$$(\Delta + \omega^2)p_B = q\ e^{-i\omega e\cdot x} .$$

Hence introducing the fundamental outgoing solution, a Hankel function, we have up to a constant

$$p_B(x) = \int H(\omega|x-x'|)\ q(x')\ e^{-i\omega e\cdot x'}\ |dx'| .$$

For large distances $e^{i\omega|x|}|x|^{1/2}p_B(x)$ tends in the direction $-e$ to

$$\int \omega^{-1/2} \exp i\omega(-e|x|+x') - e\cdot x' - |x|)\ q(x')\ dx'$$
$$\to \omega^{-1/2} \int e^{-2i\omega e\cdot x'}\ q(x')\ dx' ,$$

up to a phase factor. Thus setting $q = q_B$ to denote the Born approximation

(1.4a) $$s(e,\omega) = c\ \omega^{-1/2}\hat{q}_B(-2\omega e) .$$

where $\hat{q}_B$ is the Fourier transform of q and c is some numerical factor. Thus the relation between the approximate potential and the back scattered field can be read off by the Fourier transform. If q_B lies in Q we note that s is bounded but more than that is required of s to determine q. It is reasonable to suppose that if the scattered field is small in a suitable norm such as that q_B is small then the exact q can be found as a convergent series in powers of q_B. This has been done for 3D by Prosser in a series of papers [2]. See also Newton [3] where the full scattered field is involved and Fadeev [4].

The use of the Radon transform approach in 3D is analyzed by Yagle et al. [5]. Here we have transformed away from the Radon transform which is used in [1].

The approach used here utilizes the underlying formulation of [1] and regards the given data as $q_B(x)$. Then

(1.4) $q_B(x) = q(x) + c \int |\omega|\ p_s(e,\omega,x')\ q(x')\ e^{i\omega e\cdot(2x-x')}\ |dx'|\ d\omega\ d\theta$

where c here and elsewhere is a generic real constant and $e = (\cos\theta, \sin\theta)$. The derivation is in appendix 1.

The formal expansion is

(1.5) $$q(x) = \sum_{1}^{\infty} q_n(x)\ ,$$

where

(1.6) $$q_1(x) = q_B(x)$$
$$q_n(x) = c \int |\omega|\ p_n(e,\omega,x')\ q(x')\ e^{i\omega e\cdot(2x-x')}\ d\omega\ d\theta\ .$$

and

(1.7) $$\Delta p_n + \omega^2 p_n = q p_{n-1}$$
$$p_0 = e^{-i\omega e\cdot x}\ .$$

We would like to prove that (1.5 - 1.7) lead to a unique solution for q ϵ Q in terms of q_B ϵ Q.

Certain aspects of this problem are more difficult than in the 3D problem. This is most easily seen from the corresponding wave equation. Denoting the corresponding variables by capital letters, let

(1.8) $$P(e,t,x') = c \int e^{i\omega t} p(e,\omega,x')\, d\omega .$$

Then

(1.1*) $$P_{tt} - \Delta P + qP = 0 ,$$
$$P = 0 \text{ for } t - e \cdot x < 0 ,$$

and

$$P = \delta(t-e\cdot x) \quad \text{as} \quad t \to -\infty .$$

Then

(1.2)* $$P_{stt} - \Delta P_s + qP_s = - q\delta(t-e\cdot x)$$

(1.3)* $$S(e,t) = ic \int e^{i\omega t}\, \omega^{-1/2}\, \hat{q}_B(2\omega e)\, d\omega$$

(1.4)* $$q_B(x) = q(x) + c \int \frac{d\theta\, d\sigma}{\sigma} \frac{\partial P_s}{\partial \sigma} (e, e\cdot(2x-x') - \sigma, x')\, q(x')\, |dx'|$$

(1.6)* $$q_n = c \int \frac{dx'\, d\theta\, d\sigma}{\sigma} \frac{\partial P_n}{\partial \sigma} (e, e\cdot(2x-x')-\sigma, x')\, q(x')\, |dx'|$$

(1.7)* $$P_{ntt} - \Delta P_n = - qP_{n-1}$$

with $P_0 = \delta(t-e\cdot x)$.

The dimension problem shows up in (1.7)*. The nonhomogeneous term for $n \geq 2$ is not of compact support in time. A crude estimate suggests that there is logarithmic growth in time for P_n if there are two dimensions in space. In three dimensions, by contrast, Huyghen's principle carries the energy of P rapidly out of the main support of q and estimates are generally easier to make.

We use only some the properties of q_n that can be derived from time-formulation (*) and then use direct frequency domain estimates. define

(1.9) $$||q|| = |q|_\infty + \int |q|\, |dx|$$

and show

(1.10) $$||q_n|| \le c^n \; ||q||^n \; .$$

Thus the series (1.5) converges for $||q||$ less than some constant and can be uniquely inverted if $||q_B||$ is sufficiently small. Thus we will have proved

<u>Theorem</u>. If the norm of the Born approximate $||q_B||$ is sufficiently small there is a unique potential with the same back scattered field as the Born approximate has by linear theory.

The proof of (1.10) consists of technical (mainly calculus) estimates using iterated kernels in both the time and frequency form. In section 2 we examine q_2 which turns out to have the interesting form

(1.11) $$\int d\theta' \, d\theta \; Q(e\cdot x,e) \; Q(e'\cdot x,e') \; \delta(e\cdot e') \; ,$$

where δ is the Dirac delta function and Q is the Radon transform of q at x in the direction perpendicular to $e = (\cos \theta, \sin \theta)$. We also show that it is bounded, $||q_2|| \le c||q||^2$.

In section 3 we show that for $n > 4$, q_n is correctly bounded in L_∞; in section 4 that q_3 and q_4 are also correctly bounded in L_∞. In section 5, we show that q_n lies in L' for $n \ge 3$ again with the right bound. The proof of the theorem is then easily completed.

We shall ignore constant factors to the power n as they influence only the radius of convergence of the series. However we shall point them out where they occur.

In Appendix 1 we rederive the basic formula (1.4) and (1.4)*. In Appendix 2 we show how an iteration to achieve the solution converges using previously obtained estimates.

2. <u>The case</u> $n = 2$.

To derive the formula (1.12) we may assume q is smooth.

We have from (1.7)

$$p_1 = - \int H(\omega|x-x'|)\, q\, e^{-i\omega e \cdot x'}\, |dx'| \ .$$

Thus we find from (1.6):

$$q_2 = \int K(x,x',x")\, q(x')\, q(x")\, |dx'|\, |dx"|$$

where

$$K = \mathrm{Re}\, c \int |\omega|\, H(\omega|x'-x"|)\, e^{i\omega e \cdot (2x-x'-x")}\, d\omega \ .$$

But

$$\int_{-\pi \le \theta \le \pi} e^{i\omega e \cdot (2x-x'-x")}\, d\theta = c\, J_0(\omega|2x-x'-x"|) \ ,$$

where J_0 is the regular Bessel function of zero order. Hence we write K as $K(\rho_0, \rho_1)$ where $\rho_0 = |2x-x'-x"|$, $\rho_1 = |x'-x"|$ and

$$K = \mathrm{Re}\, c \int_{-\infty}^{+\infty} |\omega|\, J_0(\omega\rho_0)\, H(\omega\rho_1)\, d\omega \ .$$

This integral has to be properly interpreted. First we have

$$K = \mathrm{Re}\, c \int_0^{\infty} \omega\, J_0(\omega\rho_0)\, (H(\omega\rho_1) + H(-\omega\rho_1))\, d\omega \ .$$

To be outgoing, H is the Fourier transform of a function of $t-|x|$ and hence behaves like $e^{-i\omega|x|}$ at ∞. It also has a real logarithmic singularity. Hence

$$H(\varsigma) = i\, J_0(\varsigma) + N_0(\varsigma) \ , \tag{2.1}$$

and

$$K = c \int_0^{\infty} \omega\, J_0(\omega\rho_0)\, N_0(\omega\rho_1)\, d\omega \ .$$

From Bessel's equation, $(\varsigma \dot{J}_0) + \varsigma J_0 = 0$ we find $(\omega J_\omega)_\omega + \rho_0^2 \omega J = 0$ and hence $\omega JN = (\omega J_\omega N - \omega J N_\omega)_\omega$ with $\varsigma = J_0(\omega\rho_0)$, $N = N_0(\omega\rho_1)$. Hence

$$K = \frac{c}{\rho_0^2 - \rho_1^2}\, (\lim_{\Omega \to \infty} \Omega(\rho_0 \dot{J} N - \rho_1 J \dot{N}) + \frac{2}{\pi}) \ .$$

We have used the logarithmic behavior of N at $\omega = 0$ and the regularity of J. (See e.g. Courant-Hilbert Vol. 1, p. 501 for the constants.)

We now use the asymptotic behavior of the Bessel functions:

$$\Omega(\rho_0\dot{J}N - \rho_1 J\dot{N}) \sim \frac{1}{\pi}\left(-\sqrt{\frac{\rho_0}{\rho_1}}\left(\cos\Omega(\rho_0-\rho_1) - \cos\Omega(\rho_0-\rho_1-\frac{\pi}{2})\right)\right.$$

$$\left. - \sqrt{\frac{\rho_1}{\rho_0}}\left(\cos\Omega(\rho_0-\rho_1) + \cos\Omega(\rho_0-\rho_1-\frac{\pi}{2})\right)\right)$$

Hence

$$K = \lim_{\Omega\to\infty} \frac{c}{\rho_0^2-\rho_1^2}\left((1-\cos\Omega(\rho_0-\rho_1)) + (1-\frac{1}{2}\sqrt{\frac{\rho_0}{\rho_1}} - \frac{1}{2}\sqrt{\frac{\rho_1}{\rho_0}})\cos\Omega(\rho_0-\rho_1)\right.$$

$$\left. - \left(\sqrt{\frac{\rho_0}{\rho_1}} - \sqrt{\frac{\rho_1}{\rho_0}}\right)\sin\Omega(\rho_0+\rho_1)\right)$$

which we write as $\lim_{\Omega\to\infty}(K_1+K_2+K_3)$.

It is easy to see that the last term K_3 behaves in the limit like a δ-function, in fact

$$K_3 \to c(\rho_0+\rho_1)^{-1}(\rho_0\rho_1)^{-1/2}\,\delta(\rho_0+\rho_1)\ .$$

We recollect that both ρ_0 and ρ_1 are nonnegative when this factor in the kernel is applied to $q(x')q(x'')\,|dx'|\,|dx''|$. We can show there is no contribution. First, set $x = 0$ for convenience. Then $2x' = \rho_0 e + \rho_1 e'$, $2x'' = \rho_0 e - \rho_1 e'$ with $e = (\cos\theta, \sin\theta)$, $e' = (\cos\theta', \sin\theta')$. The integral becomes

$$\int \delta(\rho_0+\rho_1)(\rho_0+\rho_1)^{-1}(\rho_0\rho_1)^{1/2}\, q(\rho_0 e + \rho_1 e')\, q(\rho_0 e - \rho_1 e')\, d\rho_0\, d\rho_1\, d\theta\, d\theta'\ .$$

Set $\rho_0 = R\cos\phi$, $\rho_1 = R\sin\phi$ with $0 \le \phi \le \frac{\pi}{2}$ and the integral is the principal value of

$$c\int \delta(R\cos(\phi-\frac{\pi}{4}))\,(\cos(\phi-\frac{\pi}{4}))^{-1}\sin^{1/2}\phi\,\cos^{1/2}\phi\; q(R(e\cos\phi + e'\sin\phi))$$

$$q(R(e\cos\phi - e'\sin\phi))\,R\,dR\,d\phi\,d\theta\,d\theta'$$

which is 0 since $\cos(\phi - \frac{\pi}{4}) \neq 0$.

For K_2 we make the same change of variable and we have

$$RK_2 \; dR \; \phi = \frac{\cos^{1/2}\phi - \sin^{1/2}\phi}{\cos^{1/2}\phi + \sin^{1/2}\phi} \; \frac{\sin^{1/2}\phi - \cos^{1/2}\phi}{\cos(\phi - \frac{\pi}{4})} \quad \cos(\Omega R \cos(\phi + \frac{\pi}{4})) \; R \; dR \; d\phi \; .$$

For $R \cos(\phi + \frac{\pi}{4}) > \delta$ the integral with this measure over $q(x')q(x'')$ will tend to zero as $\Omega \to \infty$ by the Riemann Lebesgue lemma. For $R \cos(\phi + \frac{\pi}{4}) < \delta$ we note that $RK_2 dR \; d\phi$ is antisymmetric and vanishes for $\delta = 0$ so again the contribution to the integral can be shown to be arbitrarily small.

Finally we turn to K_1. The contribution to the integral is again found with the variables ρ_0, ρ_1, θ, θ'. Then we set $\rho_0 + \rho_1 = \eta$, $\rho_0 - \rho_1 = \xi$ and the integral over $q(x')q(x'') \; |dx'| \; |dx''|$ becomes

$$\int \frac{1}{\xi\eta} (1 - \cos \Omega\eta) \; q(\frac{1}{4} \eta(e+e') + \frac{1}{4} \xi(e-e')) \; q(\frac{1}{4} \eta(e-e')$$

$$- \frac{1}{4} \xi(e+\xi'))(\xi^2 - \eta^2) \; d\xi \; d\eta \; d\theta \; d\theta'$$

or with $\Omega\eta = \tau$, the integral is

$$\int \frac{1-\cos \tau}{\tau} \; d\tau \int \frac{1}{\xi} \; q(\frac{\tau}{4\Omega} (e+e') + \frac{1}{4} \xi(e-e') \; q(\frac{\zeta}{4\Omega} (e-e')$$

$$- \frac{1}{4} \xi(e+e'))(\xi^2 - \frac{\tau^2}{\Omega^2}) \; d\xi \; d\theta \; d\theta'$$

On letting $\Omega \to \infty$ we have

$$\int \frac{1-\cos \tau}{\tau} \; d\tau \int \xi \; q(\frac{1}{4} \xi(e-e')) \; q(- \frac{1}{4} \xi(e+e')) \; d\xi \; d\theta \; d\theta'$$

The first integral may be replaced by c and we are left with

$$c \int \xi \; q(\frac{1}{4} \xi(e-e')) \; q(- \frac{1}{4} \xi(e+e')) \; d\xi \; d\theta \; d\theta'$$

But $(e-e')\cdot(e+e') = 0$ so we may set $\frac{1}{4} \xi(e-e') = rE$ with $E = (\cos \phi, \sin \phi)$ and $- \frac{1}{4} \xi(e+e') = sE^{\ell}$ where $E \cdot E^{\ell} = 0$ and $r^2 + s^2 = \frac{1}{8} \xi^2$ so $\xi \; d\xi \; d\theta \; d\theta' = dr \; ds \; d\phi \; d\theta$.

We may then write the contribution to q_2 from K_1 as

$$c \int q(rE) \; q(sE^{\ell}) \; dr \; ds \; d\phi \; ,$$

which is the same as (1.11). This form will not change if we let q belong to the wider space $\underline{Q}$ provided the integration can be performed.

To consolidate we write this result as part (a) of the following lemma (shifting x back from the origin).

Lemma 2.1. If $q \in Q$ then

(a) $q_2 = c \int q(x + re)\, q(x + se^{\perp})\, dr\, ds\, d\theta$, and

(b) $||q_2|| < c||q||^2$.

Proof of (b): First for the neighborhood of r^2+s^2 we bound by the L^∞ norm. Then we set $r = R \cos \phi$, $s = R \sin \phi$ and $x = 0$; then

$$q_2 =$$

$$c\int q(R \cos \phi \cos \theta, R \cos \phi \sin \theta) q(R \sin \phi \sin \theta, -R \sin \phi \cos \theta) R\, dR\, d\phi\, d\theta$$

so

$$|q_2| \le c|q|_\infty \int_{|\cos \phi|>\delta} q(R \cos \phi \cos \theta,\ R \cos \phi \sin \theta)$$

$$\cdot \frac{R \cos \phi\ d(R \cos \phi)}{\cos^2\phi}\, d\phi\, d\theta$$

$$+ c|q|_\infty \int_{|\sin \phi|>\delta} q(-R \sin \phi \sin \theta,\ R \sin \phi \cos \theta)$$

$$\cdot \frac{R \sin \phi\ d(R \sin \phi)}{\sin^2\phi}\, d\phi\, d\theta$$

$$\le c|q|_\infty||q||_1/\delta^2 \le c||q||^2 .$$

Secondly for the L^1 estimate we calculate

$$\int |q|\ |dx| \le c \int \int |q(x+re)|\ |q(x+se^{\perp})|\ dx\, ds\, d\theta\, d\phi .$$

We let θ range from 0 to π and r and s from $-\infty$ to $+\infty$. Set $x = (e \cdot x)e + (e_1 \cdot \lambda)e_1$ and we have (for convenience dropping the absolute value signs),

$$\int q((r+e\cdot x)e + (e_\perp \cdot x)e_\perp)\ q((e\cdot x)e + (e_\perp \cdot x+s)e_\perp)\ dx_1 dx_2 dr\, ds\, d\theta .$$

Taking $x_2 = e_\perp \cdot x$, $x_1 = e \cdot x$ we have to integrate with respect to θ the integral

$$\int q(r+x_1,x_2)\ q(x_1,x_2+s)\ dx_1dx_2dr\ ds$$

But $\int q(x_1,x_2+s)\ dx_1ds = ||q||_1$ and $\int q(r+x_1,x_2)\ dx_2dr = ||q||_1$ so we have

$$\int |q_n|\ |dx| \le c||q||_1^2 .$$

For use in Appendix 2, we note, as is easily proved,

Lemma 2.2. If $q_1,q_2 \epsilon Q$, then

$$\tilde{q} = \int q_1(x + re)\ q_2(x + re^{\iota})\ dr\ ds\ d\theta$$

lies in Q and $||\tilde{q}|| \le c||q_1||\ ||q_2||$.

3. L_∞ bounds for q_n, $n > 4$.

We first derive some estimates above Hankel kernels that we use frequently.

Lemma 3.1.

(a) $$\left| \int H(\omega\rho_i)\, H(\omega\rho_{i+1})\, q(x^i)\, |dx^i| \right| \le c||q||(|\log \omega|^2 + 1),$$

and for $|\omega| > \Omega$,

(b) $$\left| \int H(\omega\rho_i)\, H(\omega\rho_{i+1})\, q(x^i)\, |dx^i| \right| < \frac{c}{|\omega|}\, ||q|| .$$

here $\rho_i = |x^i - x^{i+1}|$.

Proof: From (2.1) using the behavior of the Hankel function we have for $|\omega|\rho_i$ and $|\omega|\rho_{i+1}$ less than k say,

$$|H(\omega\rho_i)\, H(\omega\rho_{i+1})| < c(|\log \omega| + |\log \rho_i|)(|\log \omega| + |\log \rho_{i+1}|) .$$

Hence the integral of (a) over $|\omega|\rho_i$, $|\omega|\rho_{i+1} < k$ is less than

$$c \int (|\log \omega|^2 + |\log \rho_i|^2 + |\log \rho_{i+1}|^2)\, q(x^i)\, |dx^i|$$
$$\le c\, (||q||_1\, |\log \omega|^2 + ||q||_1\, |\log \delta|^2 + ||q||_\infty |\log \delta|^2\, \delta^2)$$
$$\le c||q||\, (|\log \omega|^2 + 1)$$

For $|\omega|\rho_i > k$, $|\omega|\rho_{i+1} < k$ we obtain a similar estimate since the first Hankel function is bounded. For $|\omega\rho_i|$, $|\omega\rho_{i+1}| > k$ we get the bound $||q||_1$. This completes the proof of (a).

To prove (b) we note that for $|\omega\rho_i| > k$, $|H(\omega\rho_i)| < c|\omega\rho_i|^{-1/2}$. We have

$$2\left| \int H(\omega\rho_i) H(\omega\rho_{i+1})\, q(x^i)\, |dx^i| \right| \le \int (|H^2(\omega\rho_i)|^2 + |H(\omega\rho_{i+1})|^2) |q(x')| dx'$$

We estimate the first term by $c||q||_\infty \int_0^{k|\omega|^{-1}} |H(\omega\rho_i)|^2\, \rho_i\, d\rho_i$ + $c|\omega|^{-1} \int \frac{1}{\rho_i}\, |q(x^i)|\, |dx|^i \le c||q||_\infty |\omega|^{-2} + c|\omega|^{-1}||q|| \le |\omega|^{-1}||q||$. Here we used $\left| \int \rho^{-1}\, q\, dx \right| < R||q||_\infty + R^{-1}||q||_1$ for any R. The second term is essentially the same. Thus for some Ω, $\omega > \Omega$ the inequality (b) will hold.

This lemma implies that we can obtain estimates for integrals with many products. Thus, to be sufficiently general for later purposes

Lemma 3.2. Let $\tilde{H}(\omega) = \min(|\log \omega|^2 + 1, |\omega|^{-1})$ and $\tilde{J}(\omega) = \min(1, |\omega|^{-1/2})$. Let $\tilde{q}_n = \int J(\omega\rho_0) \prod_{i=1}^{n} H(\omega\rho_i) \prod_{i=1}^{n} q^i(x^i)\, |dx^i|$. Then $|\tilde{q}_n| < c\tilde{J}(\omega)\, \tilde{H}(\omega)^{n/2} \prod_{i=1}^{n} ||q^i||$.

Proof: It follows from Lemma 3.1 that for the variables x^m, $m \neq 1$ and n that occur in pairs in the product of the Hankel functions in the integral one can estimate; $m = 1$, n are exceptions as they occur in the $J(\omega\rho_0)$ and in $H(\omega\rho_n)$. There are n-1 good terms and that gives (n-1)/2 if n is odd and (n-2)/2 if n is even. This leaves "left over q^i" terms but there is no dependent factor for n odd except those that occur in the argument $\omega\rho_0$ of the Bessel function J, namely x' and x^n. Thus the terms in q^i can be estimated by L^1 integrals except for two. The designated integral is bounded by

$$(3.1)\quad (\tilde{H}(\omega))^{(n-1)/2} \prod_{i\neq 1,n} ||q^i|| \int |J(\omega\rho_0)|\ q'(x')\ q^n(x^n)|\ |dx^1|\ |dx^n| .$$

For n even there will be one more term and the designated integral is bounded by

$$(\tilde{H}(\omega))^{(n-2)/2} \prod_{i\neq 1,2,n} ||q^i|| \int |J(\omega\rho_0)|\ ||H(\omega\rho_1)||\ q'(x')\ q^2(x^2)\ q^n(x^n)\ |dx^1|\ |dx^2|\ |dx^n|$$

But

$$\int |H(\omega\rho_1)|\ |q^2(x^2)|\ |dx^2| \leq c(|\log \omega| + 1)\ ||q^2||$$

and for $\omega > \Omega$,

$$\int |H(\omega\rho_1)|\ |q^2(x^2)|\ |dx^2| < ||q^2||_\infty \left| \int_0^{\omega^{-1}k} H(\omega\rho_1)\ \rho_1\ d\rho_1\right| + (R^{-1/2}||q^2||_1 + ||q^2||_\infty R^{3/2})|\omega|^{-1/2}$$
$$< ||q^2||_\infty (c\omega^{-2} + \omega^{-1/2}(R^{-1/2} + R^{3/2}))$$

for any R. Thus we have for n even or odd, the bound given by (3.1).

To complete the finding of the bound we can set x = 0. Then $\rho_0 = |x' + x|$. From the reglarity properties of ζ for finite argument and its behavior at ∞ we can estimate the remaining integral by $||q^1|| \; ||q^n||$ as in the previous cases. This completes the proof of Lemma 3.2.

This leads us immediately to

Lemma 3.3. Let

$$q_n^* = \int_0^\omega d\omega\, \omega\, J(\omega\rho_0)\, \left(\prod_0^{n-1} H(\omega\rho_i) + \prod_0^{n-1} H(-\omega\rho_i) \right) \prod_{i=1}^{n} q^i(x^i)\, |dx^i| \; .$$

Then $|q_n^*| \le c \prod_{i=1}^{n} ||q^i||$ for $n \ge 5$.

Proof: From the decay of $\tilde{J}$, $\tilde{H}$ as $\omega \to \infty$ we see that q_n^* satisfies the desired bound by using Lemma 3.2.

Lemma 3.4. $|q_n| \le c \prod_{i=1}^{n} ||q^i||$ for $n \ge 5$. Here q_n is defined in (1.6).

From (1.6) and (1.7) it is easily seen that $q_n = q_n^*$ if $q^i(x^i) = q(x^i)$ and we integrate over θ to obtain the Bessel function as we did in deriving q_2. Hence Lemma 3.4 follows from Lemma 3.2.

4. L_∞ bounds for $n = 3,4$.

To handle the missing cases $n = 3,4$ we must use a less crude estimate and take into account the oscillatory character of the Bessel and Hankel function. However it is much easier to carry this out for $n = 3,4$ than for $n = 2$ because the integrand is dying out, if slowly, as $\omega \to \infty$.

Lemma 4.1. For $n = 3,4$, $|q_n^*|$ is bounded by $\prod_1^n ||q^i||$ and $|q_n|$ is bounded $||q||^n$.

Proof: The integrand of q_n^* defined in Lemma 3.3 dies out respectively like $\omega^{-1/2}$, ω^{-1} for ω real and $\rho_i > \delta > 0$ say. But for ω in the complex plane with a suitable argument $\neq 0$, q_n^* dies out exponentially. We split the Bessel function $J(\omega\rho_i)$ as $H^{(1)}(\omega\rho_0) + H^{(1)}(-\omega\rho_0)$ for ω real and note that $H(\omega\rho_i) = i\, H^{(1)}(\omega\rho_i)$, see section 2. Next we note that $H^{(1)}(\omega\rho_i) \sim \exp i(\omega\rho_i - \pi/4)\, (\omega\rho_i)^{-1/2}$ for large $|\omega\rho_i|$, $|\arg\omega| < \delta$ say. Hence for $\rho_i \neq 0$ the integrand of q_n^* behaves for large $|\omega|$, up to a phase factor and a constant, like

$$\omega^{-1/2}(\rho_0\rho_1\rho_2)^{-1/2}(\exp i\omega(\rho_0+\rho_1+\rho_2) + \exp i\omega(-\rho_0+\rho_1+\rho_2) + \exp i\omega(\rho_0-\rho_1-\rho_2) + \exp i\omega(-\rho_0-\rho_1-\rho_2))$$

for $n = 3$ and a similar expression appears for $n = 4$. But the factor is $\omega^{-1}(\rho_0\rho_1\rho_2\rho_3)^{-1/2}$.

First we note that the integral over ω converges in the neighborhood of $\omega = 0$ as it did for $n \geq 5$. Without loss of generality we consider one of the exponential terms and a path of integration in the ω plane going from $\delta e^{i\alpha}$ as $\arg\omega = \alpha$ to infinity. There is no contribution from an arc at ∞ if we shift to this path. The value of α is chosen small but so that the exponential is decaying.

The easiest case is the first term for $n = 4$. We have the exponent $\rho_0+\rho_1+\rho_2+\rho_3 > 0$ so $\alpha > 0$. Set $\omega(\rho_0+\rho_1+\rho_2+\rho_3) = \Omega$. On integrating with

respect to Ω we obtain $c(\rho_0\rho_1\rho_2\rho_3)^{-1/2}$. Thus doing the same with the other terms we find

$$|q_4^*| \le c \prod_1^4 ||q^i|| + c \int (\rho_0\rho_1\rho_2\rho_3)^{-1/2} \prod_1^4 |q^i(x^i)|\ |dx^i| .$$

With $x = 0$ and bounding $(\rho_0\rho_1\rho_2\rho_3)^{-1/2}$ by $|x^1+x^4|^{-1/2}\ |x^1-x^2|^{-1/2}$ $(|x^2-x^3|^{-1} + |x^3-x^4|^{-1})$ we estimate by integrating first w.r.t. x^3 in the usual way. This gives the factor $||q^3||$. Repeat w.r.t. x^2 and again w.r.t x^4 and finally w.r.t. x^1. Thus we find

$$||q_4^*||_\infty \le c \prod_1^4 ||q^i|| .$$

To estimate q_3^* in L^∞ we use the same idea but we obtain

$$|q_3^*| \le c\pi ||q^i|| + \int (\rho_0\rho_1\rho_2)^{-1/2}(c(\rho_0+\rho_1+\rho_2)^{-1/2} + c|-\rho_0+\rho_1+\rho_2|^{-1/2}) \prod_{i=1}^3 q^i(x^i)|dx^i|$$

We estimate the first term using $(\rho_0\rho_1\rho_2)^{-1/2}(\rho_0+\rho_1+\rho_2)^{-1/2}$ is bounded by $|x^1+x^3|^{-1}(|x^1-x^2|^{-1}+|x^2-x^3|^{-1})$ and obtain, as for $n = 4$, an estimate for the first term of the form $c \prod_{i=1}^3 ||q^i||$.

The second term is more difficult because $-\rho_0+\rho_1+\rho_2$ vanishes on an ellipse if we fix x^1 and x^3. this follows from the fact that $\rho_1+\rho_2$ is the sum of the distances of x^2 to x^1 and x^3. First we use six dimensional polar coordinates to estimate the contribution from the neighborhood of $\rho_0 = \rho_1 = \rho_2 = 0$. Thus with $\rho_0^2 + \rho_1^2 + \rho_2^2 = R^2$ the contribution from $R < R_0$ to the unestimated integral in (4.1) can be estimated by

$$\Pi\ ||q^i|| \int_{R<R_0} |-\cos\phi + \sin\phi\ \sin(\theta + \tfrac{\pi}{4})|^{-1/2}\ R^5\ d\phi\ d\theta\ dR$$

where $\rho_0 = R\cos\phi$, $\rho_1 = R\sin\phi\cos\theta$, $\rho_2 = R\sin\phi\sin\theta$. The integral converges and we have a bound with the factor R_0^6.

To estimate for $R > R_0$ we first integrate over x" in the region $|-\rho_0+\rho_1+\rho_2| > \delta$ and obtain using L^1 bounds n q^i the term

$$c||q^2||\delta^{-1/2} \int \rho_0^{-1/2}(q^1(x^1)|\ |q^3(x^3)|\ |dx^1|\ |dx^3| ,$$

since

$$|\int (\rho_1\rho_2)^{-1/2}\, q^2(x^2)\, |dx^2| < \int (\rho_1^{-1}+\rho_2^{-1})\, |q^2(x^2)|\, dx^2|$$
$$< c||q^2|| .$$

Estimating the remaining integrals we obtain

$$c\, \delta^{-1/2}\, \pi||q^i|| .$$

Over the neighborhood of the ellipse $-\rho_0+\rho_1+\rho_2 = 0$ i.e. $|-\rho_0+\rho_1+\rho_2| \le \delta$ we use L^∞ estimates for q^i and estimate the area by the width δ times the length of the ellipse L. Thus we obtain from the integration w.r.t. x^2,

$$c\, R_0^{-1}||q^2|| \int \rho_0^{-1/2}\, \delta^{1/2}\, L\, |q^1|\, |q^3|\, |dx^1|\, |dx^3| ,$$

where L is the length of the ellipse with foci at x^1, x^3 and with $|x^1-x^2| + |x^2-x^3| = |x^1+x^3|$. Thus L is bounded by $|x^1+x^3| = \rho_0$. Choosing $\delta = |x^1+x^3|^{-1}$ we finally have the bound for arbitrary R_0 for the last part:

$$cR_0^{-1}||q^2||\ ||q^1||_\infty||q^3||_\infty < cR_0^{-1}\Pi\ ||q^i|| .$$

We have completed the proof of the first part of Lemma 4.1. The second part follows from the definition of q_n in (1.6) and (1.7).

5. L^1 estimates for $n \geq 3$.

We must now take into account the sign of q. First we look at the multiple kernel that occurs in q_n^*. It is

$$K = \int_0^\infty \omega\, J(\omega\rho_0)\, \left(\prod_{i=1}^{n-1} H(\omega\rho_i) + \prod_{i=1}^{n-1} H(-\omega\rho_i) \right) d\omega .$$

For $n \geq 3$ the integral converges improperly. We claim

Lemma 5.1. $K(-1)^n \geq 0$.

Proof: We have $H(\omega\rho) = -c_+ \int_{t>\rho} e^{-i\omega t}(t^2-\rho^2)^{-1/2}\, dt$ where $c_+ > 0$. Also $iJ + N = H$ so that $J = c_+ \int_{t>\rho} \sin \omega t (t^2-\rho^2)^{-1/2}\, dt$. Thus we may write

$$K = (-1)^{n-1} \int_0^\infty \int_{t_i>\rho_i} \omega \sin \omega t_0 \left(\prod_1^{n-1} e^{-i\omega t_i} + \prod_1^{n-1} e^{i\omega t_i} \right) \prod_0^n (t_i^2-\rho_i^2)^{-1/2}\, dt_i\, d\omega .$$

The integral with respect to ω would be

$$-i \int_0^\infty \left(e^{i\omega(t_0-\Sigma t_i)} - e^{i\omega(-t_0-\Sigma t_i)} + e^{i\omega(t_0+\Sigma t_i)} - e^{i\omega(-t_0+\Sigma t_i)}\right)\omega\, d\omega .$$

But

$$\mathrm{Re} \int_0^\infty i\omega e^{i\omega\sigma}\, d\omega = \mathrm{Re} \frac{\partial}{\partial\sigma} \int_0^\infty e^{i\omega\sigma}\, d\omega = \delta'(\sigma).$$

Hence

$$K = (-1)^n C_+ \int [\delta'(t_0 - \Sigma t_i) - \delta'(-t_0-\Sigma t_i) + \delta'(t_0+ \Sigma t_i)$$

$$- \delta'(-t_0+ \Sigma t_i)] \prod_0^n (t_i^2 - \rho_i^2)^{-1/2}\, dt_i \quad \text{for} \quad t_0 > \rho_0,\ t_i > \rho_i .$$

$$= (-1)^n c_+ (\delta'(t_0 - \Sigma t_i) - \delta'(-t_0 - \Sigma t_i)) \prod_0^n (t_i^2-\rho_i^2)^{-1/2}\, dt_i$$

$$\text{for } t_0 > \rho_0,\quad t_i > \rho_i .$$

There is no contribution from the second term. Integrating by parts with respect to t_0 we find

$$K = (-1)^n C_+ \int_{t_i>\rho_i>0} t_0(t_0^2-\rho_0^2)^{-3/2} \prod_1^n (t_i^2-\rho_i^2)^{-1/2}\, dt_i .$$

where $t_0 = \Sigma t_i$ since $t_0 < 0$ is inadmissible. Hence the kernel has a sign that depends only on n.

<u>Corollary to Lemma 5.1</u>: q_n alternates in sign.

If now we write $q^i = q^i_+ - q^i_-$ where + (-) indicates the positive (and negative) part we have both q^i_+ and q^i_- lying in Q if q^i does. Then

$$q^*_n = \int K \prod_1^n q^i(x^i)\, dx^i$$

and

$$\int |q^*_n|\ |dx| \le \int K \prod_1^n (q^i_+ + q^i_-)\, dx^i$$
$$= \Sigma \int K \Pi\, q^i_\pm\, dx^i \ ,$$

where Σ means over all possible choices of sign. Hence we have:

<u>Lemma 5.2</u>. q^*_n is bounded in L^1 by $2^n \prod_1^n ||q^i||$ if $\int K \Pi\, |q^i|\, dx^i$ is bounded in L^1 by $\Pi\, ||q^i||$ if $q^i > 0$.

<u>Lemma 5.3</u>. $\int K \Pi\, q^i\, dx^i$ is bounded in L^1 by $\Pi\, ||q^i||$ if $q^i > 0$, n ≥ ;

Proof: Here

$$K = \int_0^\infty \omega\, J(\omega\rho_0)\ \left(\prod_{i=1}^n H(\omega\rho_i) + \prod_{i=1}^n H(-\omega\rho_i) \right) d\omega \ .$$

To obtain the L^1 estimate we may integrate $J(\omega\rho_0)$ with respect to ξ i.e. w.r.t. $\rho_0 = |2x - x^1 - x^n|$. We use $\omega J(\omega\rho_0)\ |dx| = \omega J(\omega\rho_0)\rho_0\ d\rho_0\ d\theta_0 =$ $\omega \dot{J}(\omega\rho_0)\rho_0)^{\bullet}\ d\rho_0\ d\theta_0$.

We anticipate that the integral converges very weakly so we consider

$$||q^*_n|| = \mathbb{L}_\infty = \lim_{L\to\infty} \mathbb{L}(L) \ .$$

where

$$\mathbb{L} = \int_{L^{-\epsilon}}^\infty d\omega \int_{\substack{\rho_i<L \\ \rho_0<L}} \underline{K} \prod_1^n q^i(x^i)\ |dx^i|\ |dx| \ .$$

where

$$\underline{K} = \int_{L^{-\epsilon}}^\infty d\omega \int_{|\rho_i|<L} L\dot{J}(\omega L)\ \left(\prod_1^n H(\omega\rho_i) + \prod_1^n H(-\omega\rho_i)\right) \prod_1^n q^i(x^i)\ |dx^i|$$

Split the integral over ω into the part from $L^{-\epsilon}$ to Ω_0 and frm Ω_0 to ∞. Consider the second part first and integrate by parts noting that $L\dot{J}(\omega L)\, d\omega = dJ(\omega L)$. Thus, taking only one product, since the other one is essentially the same,

$$\int_{\Omega_0}^{\infty} L\dot{J}(\omega L)\, \Pi\, H(\omega\rho_i)\, d\omega = -J(\Omega_0 L)\, \Pi\, H(\Omega_0\rho_i) - \int_{\Omega_0}^{\infty} J(\omega L) \sum_i \rho_i \dot{H}(\omega\rho_i) \prod_{j\neq i} H(\omega\rho_i) d\omega .$$

The first term is bounded for $\Omega_0\rho_i < k$, $L^{-1/2}\Omega_0^{-1/2}\Pi\ (|\log \Omega_0| + |\log \rho_i|)$. If we integrate over $\Pi\ q^i\ dx^i$ and use $\int |\log \rho_i|\ q^i\ dx^i < ||q^i||$ and over $\Pi\ q^i\ dx^i$ where ρ_i > const. and $H(\Omega_0\rho_i)$ is bounded we find that the integrated term dies out at least like $L^{-1/2}\ \Pi\ ||q^i||$. Thus we are reduced to bounding for large L,

$$K_{\Omega_0} = \int_{\Omega_0}^{\infty} J(\omega L) \sum_i \rho_i\ \dot{H}(\omega\rho_i) \prod_{j\neq i} H(\omega\rho_i)\ d\omega ,$$

applied to $\Pi\ q^i\ dx^i$.

Collapsing the integral using Lemma 3.1 we find a typical term ($i = 1$) bounded by

$$\prod_{i=2}^{n} ||q^i|| \int_{\Omega_0}^{\infty} |J(\omega L)|\ \rho_1 |\dot{H}(\omega\rho_1)\ q^1(x^1)\ |dx^1|\ \bar{H}(\omega)^{(n-2)/2}\ d\omega$$

But

$$\int_{\rho_1 < L} \rho_1 |\dot{H}(\omega\rho_1)| q^1(x^1)\ |dx^1| < ||q^1||\omega^{-1} + \int_{\omega\rho_1 > k} (\omega\rho_1)^{-1/2}\ \rho_1\ q^1\ |dx^1|$$

$$< ||q^1||\ (\omega^{-1} + L^{1/2}\ \omega^{-1/2}) .$$

Hence for large L the bound for the integral from Ω_0 to ∞ is

$$\Pi\ ||q^i|| \int_{\Omega_0}^{\infty} (\omega L)^{-1/2}\ (\omega^{-1} + L^{1/2}\ \omega^{-1/2})\ \bar{H}(\omega)^{\frac{n-2}{2}}\ d\omega$$

$$< \Pi\ ||q^i||\ (L^{-1/2} \log \Omega_0^{\frac{n-2}{2}}\ \Omega_0^{-1/2} + \text{const.}) \quad \text{for } n \geq 3$$

$$\leq c\ \Pi\ ||q^i|| \quad \text{for } n \geq 3 .$$

Note that this term does not go to zero as $L \to \infty$.

The same estimate holds for the remaining term.

We are left with the integral from $L^{-\epsilon}$ to Ω_0. This is the trickiest and reflects the poor decay properties of the time dependent problem, see section 1.

For this integral we can consider L sufficiently large so that $J(\omega L) \sim (\omega L)^{-1/2} \cos(\omega L - \frac{\pi}{4})$. Setting $\omega L = \Omega$, the kernel, which we shall show tends to 0 is

$$\int_{L^{1-\epsilon}}^{\Omega_0 L} \Omega^{-1/2} \sin(\Omega - \tfrac{\pi}{4}) \left(\Pi\, H(\Omega\sigma_i) + \Pi\, H(-\Omega\sigma_i)\right) d\Omega ,$$

where $\sigma_i = \rho_i/L < 1$.

If $\sigma_i > \delta$ for at least two values of i the limit of this integral as $L \to \infty$ would be zero because of the decay of the Hankel functions.

We consider the integral in the complex Ω plane on a path chosen so that the integral is decaying exponentially. Thus a typical term is the integral of $\Omega^{-1/2}\exp\ i(\Omega-\pi/4)\ \Pi\, H(\Omega\sigma_i)$ on $|\Omega| = L^{\epsilon}$, $\Omega = e^{i\alpha}|\Omega|$ and $|\Omega| = \Omega_0 L$. The integral on the ray for $\alpha \neq 0$ decays exponentially and is bounded by

$$\int_{L^{1-\epsilon}}^{\Omega_0 L} |\Omega|^{-1/2}\, e^{-\alpha|\Omega|}\ \Pi\ |H(|\Omega|\sigma_i\, e^{i\alpha})|\ d|\Omega|$$

$$< c \int |\Omega|^{-1/2}\, e^{-\alpha|\Omega|}\ \Pi\ (|\log \Omega\sigma_i|+1)\ d|\Omega|$$

$$< c\ L^{-(1-\epsilon)/2}\, e^{-\alpha L^{1-\epsilon}}\ \alpha^{-1}(|\log L^{-\epsilon}| + \log \rho_i) .$$

Integrated with respect to $\Pi\ q^i(x^i)\ dx^i$ we find that this term $\to 0$ as $L \to 0$ for $||q^i||$ bounded.

The integral on $|\Omega| = L^{\epsilon}$ is bounded by

$$c\ \ell^{1/2} \int_0^{\alpha} e^{-\ell \sin\theta}\ \Pi\ H(\ell\sigma_i\, e^{i\theta})\ e^{i\theta}\ d\theta ,$$

with $\ell = L^{1-\epsilon}$ or for small α by

$$c\ell^{1/2} \int_0^{\alpha} e^{-\ell\theta} \Pi (|\log L^{\epsilon}| + |\log \rho_i|) \, d\theta \ ,$$

or

$$c\ell^{-1/2} \Pi (|\log L^{\epsilon}| + |\log \rho_i|) \ .$$

Applied to the integral over $\Pi\, q^i(x^i)\, dx^i$ we obtain the bound $\ell^{-1/2} |\log L^{\epsilon}|^n \ \Pi \ ||q^i|| = L^{-\frac{1}{2}(1-\epsilon)} |\log L^{\epsilon}|^n \ \Pi \ ||q^i||$. Letting $L \to \infty$ this term $\to 0$. The last term is treated in the same way.

This completes the proof of Lemma 5.2.

<u>Proof of Theorem</u>: From Lemma 5.2, Lemma 5.1 follows. Combined with Lemmas 2.1 (b) for $n = 2$, Lemma 3.4 for $n \geq 5$ and Lemma 4.1 for $n = 3,4$, we have proved (1.10).

From (1.5) and (1.6) we have

$q(x) + Tq = q_B$,

where Tq is a convergent sum for $||q||$ sufficiently small, bounded by $||q||^2$. Hence if $q_B \epsilon Q$ and is sufficiently small this equation can be solved for q uniquely.

Appendix 1. Derivatin of (1.4) and (1.4)*.

To derive (1.4)* we use the integral equatin derived from (1.2)* along with $P_s = 0$ by using the 2D Riemann function. Again c is a given constant.

(A1.1) $$P_s = c \int ((t-t')^2 - |x-x'|^2)^{-1/2} (\delta(t'-e\cdot x') + P_s(e,t',x'))\, q(x')\, |dx'|dt'$$

The far field in the back scattered direction -e is given by $S(\tau,e)$ and is obtained from P_s by setting $t-|x| = \tau$, $x = -e|x|$, $t+|x| \to \infty$ which yields $c \int (\tau-t'-e\cdot x')^{-1/2} q(x') (\delta(t'-e\cdot x') + P_s(e,t',x'))\, |dx'|\, dt$ for $S(\tau,e)$ or

$$c \int (\tau-\sigma)^{-1/2} q(x') (\delta(\sigma-2e\cdot x') + P_s(e,\sigma-e\cdot x',x')\, d\sigma\, |dx'| .$$

This is the Abel transform of

$$\int q(x')(\delta(\sigma-2e\cdot x') + P_s(e,\sigma,e\cdot x',x'))\, |dx'| .$$

So we may regard this quantity as given by the back scattered field. The first term is of lower order in q and is also a Radon transform of q. Hence if we regard the Born approximation as given R.T. q_B = R.T. $q + \int q(x')\, P_s(e,\sigma-e\cdot x',x')\, |dx'|$. Inverting the Radon transform we obtain (1.4)*. To obtain (1.4) we insert (1.8) and integrate with respect to σ. Noting that $\int_{-\infty}^{+\infty} \frac{e^{i\omega\sigma}}{\sigma}\, d\sigma$ changes sign with ω for ω real but is independent of ω for $\omega > 0$ or $\omega < 0$ we easily obtain (1.4).

Appendix 2. Convergence of an Iterative Procedure.

From

$$q + Tq = q_B$$

we may use an iterative or power series expansion to find its solution. In [1] we have attempted the iteration. Here we show the iteration converges for $||q_B||$ small. The method is to find q_N at the N^{th} iteration from

$$q_N = q_B - T\, q_{N-1} .$$

Thus $q_N - q_{N-1} = -Tq_{N-1} + Tq_{N-2}$.

Let $Q_N = \max_{J \le N-1} ||q_J||$, i.e. the max norm that occurs up to and including the N^{th} iteration. Then by lemmas 2.2, 3.3, 4.1 and 5.2 if we define q_{nN} in the obvious way,

$$||q_{nN} - q_{nN-1}|| \le c^{n-1} n\, Q_{N-1}^{n-1} ||q_{N-1} - q_{N-2}|| .$$

Hence

$$||q_N - q_{N-1}|| \le \frac{cQ_{N-1}}{(1-cQ_{N-1})^2}\, ||q_{N-1} - q_{N-2}|| .$$

We also have with $q_{1N} = q_B$

$$||q_{nN}|| \le c^n Q_{N-1}^n ,$$

so that

$$||q_N|| \le ||q_B|| + \frac{c^2 Q_{N-1}^2}{1-cQ_{N-1}}$$

or

$$Q_N \le Q_1 + \frac{c^2 Q_{N-1}^2}{1-cQ_{N-1}} .$$

We claim if $Q_1 = ||q_B||$ is sufficiently small the iteration converges. To converge we need, setting $cQ_N = X_N$

$$\frac{X_N}{(1-X_N)^2} \ll 1 .$$

Claim. If $x_1 < 1/16c$ then $x_N < 1/2c$. We may assume $c > 10$ say.

Proof: By the inequality for Q_N we have if $x_{N-1} < 1/2c$,

$$x_N \le x_1 + \frac{c\,x_{N-1}^2}{1-x_{N-1}} \le \frac{1}{16c} + \frac{1}{4c(1-\frac{1}{2c})}$$

$$= \frac{1}{2c}\left(\frac{1}{8} + \frac{c}{2(2c-1)}\right) .$$

Hence $x_N < \frac{1}{2c}$ provides $\frac{c}{(2c-1)} < \frac{7}{4}$ or $4c < 14c-7$, or $c > 7/10$.

this guarantees that $\frac{x_N}{(1-x_N)^2} < 1$. Hence the iteration converges.

References

[1] Bayliss, A., Lin, Y., Morawetz, C. S., preprint.

[2] Prosser, R. T., J. Math. Phys. 10 (10) Oct. 1969, 17(10) Oct. 1978, 21(11) Nov. 1980,, 23(11) Nov. 1982.

[3] Newton, Roger G., J. Math. Phys. 22(3), Mar 1981.

[4] Fadeev, L. D., Itogi Nauk Tekh. Sov. Probl. Mat. 3 93 (1974) J. Sov. Math. 5, 334 (1976).

[5] Yagle, A. E., J. Math. Phys. 27 (10) Oct. 1986.

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